Original article

Statistical Distributions of Crests and Trough of Sea Surface Waves

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Abstract

In many practical applications, a statistical description of waves is needed to calculate and predict their impact on ships, coastal structures and beaches. This paper investigates the statistics of the trough *Th* and the crest *Cr* of sea surface waves in the coastal zone of the Black Sea. The analysis uses data from direct wave measurements obtained on a stationary oceanographic platform of the Marine Hydrophysical Institute of the Russian Academy of Sciences. In all situations, the mode of the *Th* and *Cr* distributions is shifted to the region of higher values relative to the Rayleigh distribution mode. As a rule, the analysis of the distributions of trough and crest is carried out within a second-order nonlinear model based on the Stokes wave. It is shown that within the framework of this model it is possible to describe only the average distribution over an ensemble of situations, while for practical tasks it is necessary to know the deviations from these values. The type of *Th* and *Cr* distributions significantly depends on the skewness of the distribution of sea surface elevations A_η. With $A_\eta < 0$, the probability density function *Th* and *Cr* are almost identical. The second-order nonlinear model, in which the condition $A_\eta > 0$ is always fulfilled, does not describe this situation. The probability density functions *Th* and *Cr* obtained with $A_\eta > 0$ correspond qualitatively to this model. Changes in the excess kurtosis of the distribution of sea surface elevations have a lesser effect on the probability density functions *Th* and *Cr*.

Keywords*:* sea surface, waves, trough, crest, statistical distributions, Black Sea

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Статистические распределения высоты гребней и глубины впадин морских поверхностных волн

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Аннотация
В настоящей работе исследуются статистические распределения глубины впадин *Th* и высоты гребней *Cr* морских поверхностных волн в прибрежной зоне Черного моря. Для анализа используются данные прямых волновых измерений, полученные на стационарной океанографической платформе Морского гидрофизического института РАН. Во всех ситуациях мода распределений *Th* и *Cr* смещена в область более высоких значений относительно моды распределения Рэлея. Как правило, анализ распределений глубин впадин и высот гребней проводится в рамках нелинейной модели второго порядка, построенной на основе волны Стокса. Показано, что в рамках указанной модели можно описать только средние по ансамблю ситуаций распределения, в то время как для практических задач необходимо знать отклонения от этих значений. Вид распределений *Th* и *Cr* существенно зависит от асимметрии распределения возвышений морской поверхности A_η. При $A_\eta < 0$ функции плотности вероятностей *Th* и *Cr* почти совпадают. Нелинейная модель второго порядка, в рамках которой всегда выполняется условие $A_\eta > 0$, не описывает эту ситуацию. Полученные при $A_\eta > 0$ функции плотности вероятностей *Th* и *Cr* качественно соответствуют данной модели. Изменения эксцесса распределения возвышений морской поверхности в меньшей мере влияют на функции плотности вероятностей *Th* и *Cr*.

Ключевые слова: морская поверхность, волны, впадина, гребень, статистические распределения, Черное море

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Introduction

The study of sea wave statistical distributions and the identification of rogue waves are among the urgent tasks of modern oceanology [1]. In a linear wave field, which represents a superposition of sinusoidal waves with random phase, provided that the wave spectrum is sufficiently narrow, the distribution of wave heights is described by the Rayleigh distribution [2]. It also describes the distributions of crest heights and trough depths [3]. The Rayleigh distribution is generally regarded as the lower limit giving the lowest probabilities for rogue waves [4]. The linear model also underestimates strongly the probability of high crests [5].

Crest *Cr* refers to the maximum value of wave record $\eta(t)$ between the time it crosses the zero level from bottom to top and the time it crosses that level from top to bottom [6]. Similarly, trough *Th* is minimum value $\eta(t)$ between two consecutive crossings of the zero level from top to bottom and from bottom to top. Wave height is defined as the sum of consecutive maximum and minimum values between two points where the wave record crosses $\eta(t)$ the zero level upwards or downwards, i. e., $H = Cr + Th$ [7].

Deviations of sea wave statistical distributions from the linear model are usually described within a second-order nonlinear model which is based on the decomposition of the wave profile into small parameter powers (steepness) [8]. In the above model, the skewness of the sea surface elevation distribution is always higher than zero [9], the crests are higher and the troughs are shallower than predicted by linear theory [10]. Both of these conditions are not always fulfilled in marine environments. Measurements made in different areas of the world ocean have shown that the lower limit of the range where the skewness varies lies in the region of negative values [11, 12]. The second-order nonlinear model describes only average trends of skewness and excess kurtosis changes not allowing to describe the whole variety of situations occurring in the sea [13].

The ratios of crest and trough vary widely. Some situations are observed when the maximum trough is greater than the maximum crest during a measurement session [14, 15]. According to measurements in the Black Sea, the probability of an event in which the trough of the highest wave in the measurement session is greater than its crest reaches 10% [16].

Less attention has been paid to the analysis of the distribution of sea wave troughs than to the statistical description of their crests, although the distribution of troughs is of great importance for a number of engineering applications [9]. The purpose of this paper is to analyse jointly the distributions of crests and troughs of surface waves.

Measurement equipment and conditions

Wave measurement data obtained on the stationary oceanographic platform of Marine Hydrophysical Institute of the Russian Academy of Sciences were used to study the statistical characteristics of surface waves. The stationary oceanographic platform is located in the coastal part of the Black Sea off the Southern Coast of Crimea at a depth of about 30 metres. Two types of wave recorders were used to measure surface waves. The wave recorders of the first type contain a vertically stretched nichrome string as a sensor [17] and the nichrome string is coiled with a constant pitch on a vertically oriented supporting cable-tether in wave recorders of the second type [18].

This paper analyses the measurement data obtained in the summer and autumn of 2006 as well as in the winter of 2018. In 2006, measurements were taken in sessions lasting several hours; in 2018, wave measurements were taken continuously for a month. Continuous recordings of sea surface elevations were divided into 20 min fragments. A total of 2380 twenty-minute fragments were used for the analysis.

For each fragment, crest Cr and trough Th of separate waves were determined and significant wave height H_S, skewness A and excess kurtosis E of surface elevation were calculated. Only waves satisfying the conditions of $Cr > 5$ cm and $Th > 5$ cm were considered in the analysis. Hereinafter, parameter Th is equal to the trough modulus.

Wave measurements carried out in different seasons made it possible to cover a wide range of meteorological parameters. The average wind speed during the measurement session varied from conditional zero (the threshold of propeller starting) to 26 m/s. The wind speed reached 35 m/s in gusts. The wave periods calculated from the maximum of the wave spectrum were in the range from 1.1 s to 9 s. Significant wave height varied from 0.1 m to 2.3 m. The values of wave steepness (nonlinearity parameter) ranged mainly from 0.009 to 0.09.

Distributions of troughs and crests

For statistical moments $\eta(t)$, we introduce the following notation

$$\mu_n = \langle \eta^n(t) \rangle ,$$

where $\langle \ldots \rangle$ is averaging. Let us assume that the average value of a random variable is $\mu_1 = 0$, then the skewness and excess kurtosis of the surface elevation distribution are equal to $A_\eta = \mu_3/\mu_2^{3/2}$ and $E_\eta = \mu_4/\mu_2^2 - 3$, respectively.

To compare the statistical distributions of troughs and crests determined in different situations, we will use normalised wave records

$$\tilde{\eta}(t) = \eta(t)/H_S , \qquad (1)$$

where H_S is significant wave height connected to the second statistical moment of sea surface elevations by relation $H_S = 4\sqrt{\mu_2}$.

The probability density function of the Rayleigh distribution describing distributions of Cr and Th under the linear model is as follows

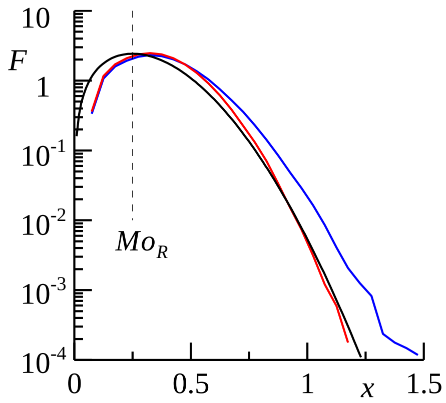
$$F_R(x) = \frac{x}{a^2} \exp\left(-\frac{x^2}{2a^2} \right), \quad x \geq 0 , \qquad (2)$$

where $a = H_S / 4$. Considering (1), we obtain that in our case mode of distribution (2) is defined as $Mo_R = 0.25$.

Empirical probability density functions of Cr and Th were calculated based on histograms constructed with equal intervals of 0.05. Fig. 1 shows the empirical probability density functions of crests $F_{Cr}(x)$ and troughs $F_{Th}(x)$ calculated over the entire measurement data set. It can be seen that modes of empirical distributions Mo_{Cr} and Mo_{Th} are shifted relative to the Rayleigh distribution mode towards higher values x, i. e., the following conditions are fulfilled

$$Mo_{Cr} > Mo_R , \quad Mo_{Th} > Mo_R .$$

The modes of the Cr and Th distributions are located in neighbouring intervals, with their centers $Mo_{Cr} = 0.375$ and $Mo_{Th} = 0.325$.



F i g . 1 . Probability density functions F (average over an ensemble of situations). The blue curve is $F_{Cr}(x)$, the red curve is $F_{Th}(x)$, the black curve is $F_R(x)$

The consideration of nonlinearity leads to the fact that the probability of high crests becomes greater than in the linear model, in which this probability is described by the Rayleigh distribution, and the probability of deep troughs is smaller [9]. It follows from Fig. 1 that inequality $F_{Cr}(x) > F_R(x)$ is true in region $x > Mo_{Cr}$. The ratio between $F_{Th}(x)$ and $F_R(x)$ changes at $x_0 \approx 0.8$, inequality $F_{Th}(x) > F_R(x)$ takes place in region $x_0 > x > Mo_{Cr}$, reciprocal ratio $F_{Th}(x) < F_R(x)$ takes place at $x > x_0$. Thus, in the area of high crests and deep troughs, deviations $F_{Cr}(x)$ and $F_{Th}(x)$ from $F_R(x)$ occur in the direction predicted by the second-order nonlinear model [19], i. e., the average distribution of crests and troughs over an ensemble of situations corresponds to this model qualitatively.

Earlier studies of senior statistical moments of sea surface elevations have shown that the second-order nonlinear model makes it possible to describe only average trends of skewness and excess kurtosis, but does not allow describing the whole variety of situations occurring under sea conditions [13]. The skewness and excess kurtosis values vary over a much wider range than the model suggests. In particular, the model estimates of surface elevation distribution skewness A_η and excess kurtosis E_η are always positive [20], while situations in which $A_\eta < 0$ and/or $E_\eta < 0$ are often observed under sea conditions [12]. Fig. 2 shows at what values of A_η and E_η the wave records analysed in this paper were obtained.

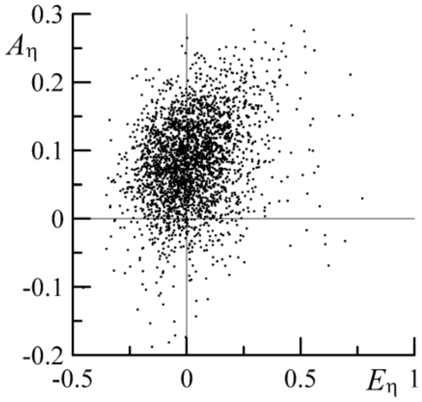


Fig. 2. Changes in the skewness A_η and excess kurtosis E_η of sea surface elevations

Effect of skewness A_η and excess kurtosis E_η

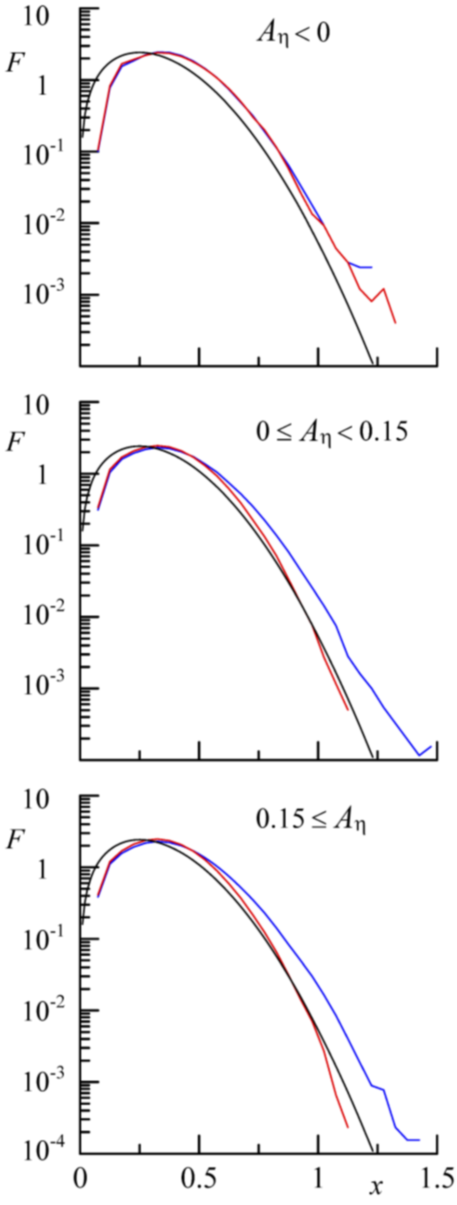
Usually, the distributions of crests and troughs are analysed within a second-order nonlinear model based on the decomposition of the wave profile into a series of small parameter powers [19–22]. In [8], a simplified second-order nonlinear model, which is the sum of linear $\eta_L(x, t)$ and nonlinear $\eta_N(x, t)$ components, is proposed to describe the sea surface statistical characteristics. The model is constructed for waves propagating in deep water in the narrowband spectrum approximation. It is described by the amplitude-modulated Stokes wave equation with average frequency ω and random phase ε

$$\eta(x,t) = \eta_L(x,t) + \eta_N(x,t) = a_r(x,t)\cos\theta + \frac{1}{2}k_p a_r^2(x,t)\cos(2\theta), \qquad (3)$$

where $a_r(x, t)$ is envelope; $\theta = k_p x - \omega t + \varepsilon$; k_p is wave number corresponding to the wave spectrum peak. The local maxima of nonlinear term $\eta_N(x, t)$ coincide with the crest and trough of the linear wave $\eta_L(x, t)$, hence, the ridge and trough within model (3) are equal [9]

$$Cr_N = a_r + \frac{1}{2}k_p a_r^2 , \quad Th_N = a_r - \frac{1}{2}k_p a_r^2 .$$

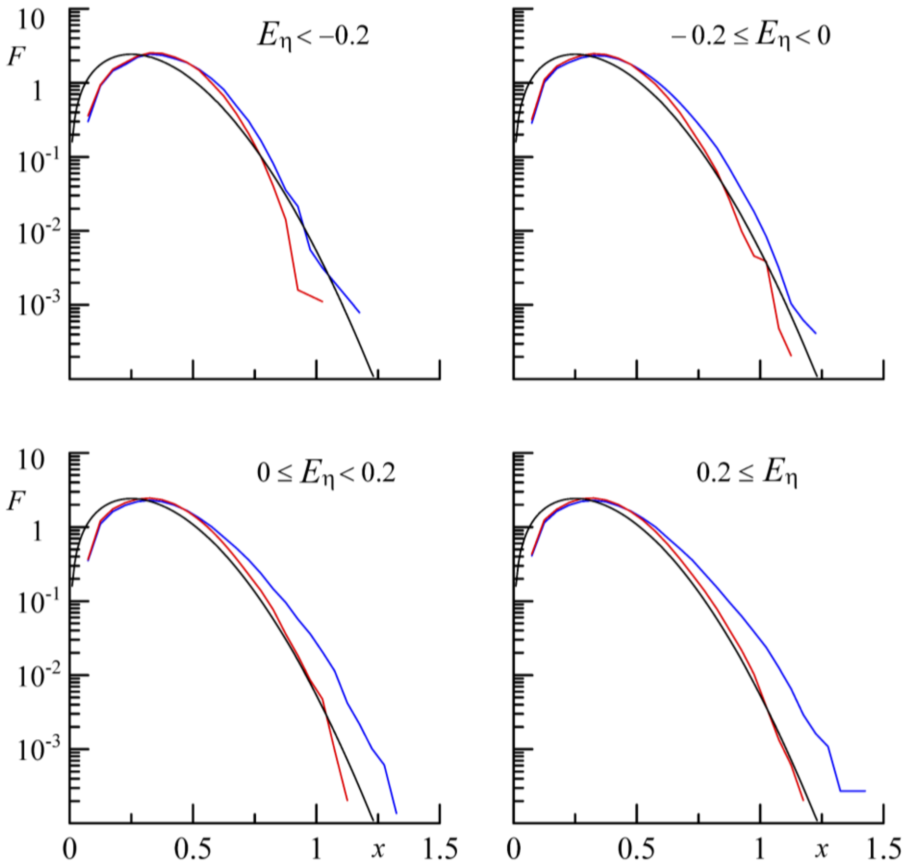
In order to assess how applicable this model is to the description of statistical distributions of crests and troughs, it is necessary to analyse how functions $F_{Cr}(x)$ and $F_{Th}(x)$ change in different situations, in particular, when the skewness or excess kurtosis is negative. Fig. 3 shows the results of this analysis.



F i g . 3 . Probability density functions F calculated for three ranges of skewness A_η. The blue curve is $F_{Cr}(x)$, the red curve is $F_{Th}(x)$, the black curve is $F_R(x)$

It follows from Fig. 3 that when changing the sign of skewness A_η, the type of function $F_{Th}(x)$ changes significantly. If condition $A_\eta < 0$ takes place, equality $F_{Th}(x) \approx F_{Cr}(x)$ is observed. Note that equality $F_{Th}(x) = F_{Cr}(x)$ takes place within the linear model when the *Cr* and *Th* distributions are described by the Rayleigh distribution. In this case, the difference from the linear model at $A_\eta < 0$ is that inequalities $F_{Cr}(x) > F_R(x)$ and $F_{Th}(x) > F_R(x)$ are fulfilled in region $x > 0.45$.

As follows from Fig. 4, changes in the excess kurtosis have a lesser effect on the type of functions $F_{Cr}(x)$ and $F_{Th}(x)$.



F i g . 4 . Probability density functions F calculated for four ranges of excess kurtosis E_η. The blue curve is $F_{Cr}(x)$, the red curve is $F_{Th}(x)$, the black curve is $F_R(x)$

Conclusion

Based on direct wave measurements carried out under sea conditions, the distributions of troughs *Th* and crests *Cr* of sea surface waves were analysed. On average over an ensemble of situations, the greater values of crests calculated from the measurement data have higher probability than the Rayleigh distribution suggests, and the probability of deep troughs is smaller. Such distributions of crests and troughs correspond to a second-order nonlinear model qualitatively.

At the same time, the second-order nonlinear model fails to describe $F_{Th}(x)$ and $F_{Cr}(x)$ when sea surface elevation distribution skewness A_η is negative. It is shown that functions $F_{Th}(x)$ and $F_{Cr}(x)$ are approximately equal at $A_\eta < 0$.

Changes in the excess kurtosis of the distribution of sea surface elevations have a lesser effect on the probability density functions of *Th* and *Cr* than changes in the skewness of the distribution.

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